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Comments on M theory dynamics on G_2 holonomy manifolds

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Abstract: We study the dynamics of M-theory on G_2 holonomy manifolds, and consider in detail the manifolds realized as the quotient of the spin bundle over $\mathbf{S}^3$ by discrete groups. We analyse, in particular, the class of quotients where the triality symmetry is broken. We study the structure of the moduli space, construct its defining equations and show that three different types of classical geometries are interpolated smoothly. We derive the $\mathcal{N} = 1$ superpotentials of M-theory on the quotients and comment on the membrane instanton physics. Finally, we turn on Wilson lines that break gauge symmetry and discuss some of the implications.

Keywords: M-Theory, Supersymmetry and Duality.

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Contents

1. Introduction

The study of the dynamics of M-theory on G_2 holonomy manifolds has been receiving much attention recently. It provides a framework for the analysis of $\mathcal{N} = 1$ supersymmetric systems in four dimensions. When the G_2 holonomy manifolds are smooth and large the low energy physics can be described by a Kaluza-Klein reduction of eleven-dimensional supergravity (see e.g. [1]). These cases are the less interesting ones since at low energy we find neither non-abelian gauge symmetry nor chiral matter. For these we have to consider singular G_2 manifolds.

Recently, there has been a significant progress in understanding the dynamics of M-theory on singular G_2 manifolds [2]–[9]. For instance, appropriate singular quotients of $S(\mathbf{S}^3)$, the spin bundle over $\mathbf{S}^3$, yield $\mathcal{N} = 1$ super Yang-Mills (SYM) theory with an ADE gauge group [10, 11]. It has been shown that M-theory on these backgrounds reproduces the vacuum structure of the field theory. It was demonstrated in [4], when studying the duality proposed in [12], that a singular quotient of $S(\mathbf{S}^3)$ can be interpolated smoothly to a non-singular quotient of $S(\mathbf{S}^3)$ using a G_2 flop. For other work on various aspects of G_2 holonomy compactifications and branes, see [13]–[48].

The dynamics of M-theory on $S(\mathbf{S}^3)$ and its quotients was discussed in detail in [5]. It was shown that the moduli space of the theory is given by a complex one-dimensional manifold that interpolates between three different $S(\mathbf{S}^3)$ smoothly. They also confirmed the result of [4] by showing that the moduli space of M-theory interpolates smoothly between a singular quotient of $S(\mathbf{S}^3)$ that yields pure SYM at low energy and a non-singular quotient of $S(\mathbf{S}^3)$.

The purpose of this paper is to extend the results of [5] to more general quotients of $S(\mathbf{S}^3)$. We will mainly analyse a "double" quotient, where the quotient group is a subgroup of the isometry of $S(\mathbf{S}^3)$. We will study the moduli space of M-theory on the quotient and show that the moduli space interpolates smoothly between three different types of classical geometries. These three geometries yield different low energy dynamics. Thus, the result can be viewed as another example of a G_2 flop. We will also compute the exact superpotentials of M-theory on the quotients of $S(\mathbf{S}^3)$, and comment on the membrane instanton physics. An interesting feature of the models that we study is the possibility of turning on Wilson lines that break gauge symmetry (see also [45]). We will discuss some of the implications.

The paper is organized as follows. In section 2 we review the geometry of $S(\mathbf{S}^3)$ and its quotients. In section 3 we will construct the complex curve that describes the moduli space of M-theory on a double quotient. We will show that the curve interpolates smoothly between three different classical geometries, and suggest a new G_2 flop. In section 4 we compute exact superpotentials of M-theory on quotients of $S(\mathbf{S}^3)$, and discuss some implications to the membrane instanton physics. In section 5 we will make some comments on M-theory on the double quotient with Wilson lines. Section 6 is devoted to a discussion of open problems.

2. The spin bundle $S(\mathbf{S}^3)$

The seven-dimensional spin bundle $S(\mathbf{S}^3)$ is asymptotic at infinity to a cone over the six-dimensional space $Y = \mathbf{S}^3 \times \mathbf{S}^3 = \mathrm{SU}(2)^3/\,\mathrm{SU}(2)$. The Einstein metric on Y can be constructed as follows [5]. Let $(g_1, g_2, g_3) \in \mathrm{SU}(2)^3$ be a group element, and impose the equivalence relation

$$(g_1, g_2, g_3) \cong (g_1 h, g_2 h, g_3 h), \quad h \in \mathrm{SU}(2)\,. \tag{2.1}$$

Define $a = g_2 g_3^{-1}, b = g_3 g_1^{-1}, c = g_1 g_2^{-1}$ $(abc = 1)$. The Einstein metric on Y is given by

$$d\Omega_Y^2 = da^2 + db^2 + dc^2\,, \tag{2.2}$$

where $da^2 = -\mathrm{tr}\,(a^{-1}da)^2$ etc.

The metric on Y admits the isometry group $\mathrm{SU}(2)_u \times \mathrm{SU}(2)_v \times \mathrm{SU}(2)_w$:

$$g_1 \to u g_1\,, \qquad g_2 \to v g_2\,, \qquad g_3 \to w g_3\,, \tag{2.3}$$

or equivalently

$$a \to v a w^{-1}\,, \qquad b \to w b u^{-1}\,, \qquad c \to u c v^{-1}\,. \tag{2.4}$$

In addition, the metric admits the triality symmetry Σ_3 generated by

$$\begin{aligned} &\alpha : (a, b, c) \to (c^{-1}, b^{-1}, a^{-1})\,, \\ &\beta : (a, b, c) \to (b, c, a)\,. \end{aligned} \tag{2.5}$$

Note that Σ_3 acts on (g_1, g_2, g_3) by permutations.

We can extend the cone on Y to the interior and obtain a seven-dimensional G_2 holonomy manifold. In fact one obtains three different G_2 holonomy manifolds which differ by the $\mathbb{Z}_2 \subset \Sigma_3$ symmetry group. They are:

$$
\begin{aligned}
X_1, X_1' : \qquad & ds^2 = \frac{dr^2}{1-(r_0/r)^3} + \frac{r^2}{72}\left(1-(r_0/r)^3\right)\left(2dc^2 - da^2 + 2db^2\right) + \frac{r^2}{24}da^2\,, \\
X_2, X_2' : \qquad & ds^2 = \frac{dr^2}{1-(r_0/r)^3} + \frac{r^2}{72}\left(1-(r_0/r)^3\right)\left(2da^2 - db^2 + 2dc^2\right) + \frac{r^2}{24}db^2\,, \\
X_3, X_3' : \qquad & ds^2 = \frac{dr^2}{1-(r_0/r)^3} + \frac{r^2}{72}\left(1-(r_0/r)^3\right)\left(2db^2 - dc^2 + 2da^2\right) + \frac{r^2}{24}dc^2\,. \qquad (2.6)
\end{aligned}
$$

Here r_0 is a constant which corresponds to the size of the base $\mathbf{S}^3$.

The $\mathbb{Z}_2$ isometry groups of X_i, $\mathbb{Z}_2^{(i)}$, act as

$$
\begin{aligned}
\mathbb{Z}_2^{(1)} &: \ (a,b,c) \to (a^{-1}, c^{-1}, b^{-1})\,, \\
\mathbb{Z}_2^{(2)} &: \ (a,b,c) \to (c^{-1}, b^{-1}, a^{-1})\,, \\
\mathbb{Z}_2^{(3)} &: \ (a,b,c) \to (b^{-1}, a^{-1}, c^{-1})\,.
\end{aligned}
\qquad (2.7)
$$

Note that $\mathrm{SU}(2)_u \times \mathrm{SU}(2)_v \times \mathrm{SU}(2)_w$ is also the isometry group of X_i.

It is useful to rewrite these metrics in the form of [11]

$$
ds^2 = \frac{dr^2}{1-(r_0/r)^3} + \frac{r^2}{9}\left(1-(r_0/r)^3\right)\left(\sigma^a - \frac{1}{2}\Sigma^a\right)^2 + \frac{r^2}{12}(\Sigma^a)^2\,. \qquad (2.8)
$$

Here σ^a, $a = 1,2,3$, are the left-invariant one-forms on $\mathbf{S}^3$ in an $\mathbb{R}^4$ fibration, and Σ^a the left-invariant one-forms on the base $\mathbf{S}^3$.

It admits the isometry

$$
\mathrm{SU}(2)_L \times \widetilde{\mathrm{SU}(2)}_L \times \mathrm{SU}(2)_R^{\mathrm{diag}}\,, \qquad (2.9)
$$

where $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ and $\widetilde{\mathrm{SU}(2)}_L \times \widetilde{\mathrm{SU}(2)}_R$ are the isometry groups for the fiber $\mathbf{S}^3$ and the base $\mathbf{S}^3$, respectively. Here $\mathrm{SU}(2)_R^{\mathrm{diag}}$ is the diagonal subgroup of $\mathrm{SU}(2)_R \times \widetilde{\mathrm{SU}(2)}_R$. The associative three-form of this G_2 holonomy manifold is given by [4]

$$
\Omega = \frac{r_0^3}{72}\,\epsilon_{abc}\,\Sigma^a\Sigma^b\Sigma^c + d\left(\frac{r^3 - r_0^3}{9}\,\Sigma^a\sigma^a\right). \qquad (2.10)
$$

This is invariant under the isometry group (2.9). Therefore, any quotient of (2.8) by a finite subgroup of the isometry group (2.9) preserves the G_2 structure.

We write the left-invariant one-forms in the form

$$
\sigma^a = -\frac{i}{2}\,\mathrm{tr}(T^a g^{-1} dg)\,, \qquad \Sigma^a = -\frac{i}{2}\,\mathrm{tr}(T^a \tilde{g}^{-1} d\tilde{g})\,, \qquad (2.11)
$$

with $g, \tilde{g}$ being the SU(2) group elements of the fiber and base $\mathbf{S}^3$, respectively. T^a are SU(2) generators with the normalization $\mathrm{tr}(T^aT^b) = 2\delta^{ab}$. We can label X_i, X_i' as

- X_1:

$$\begin{aligned} g &= b^{-1}\,, \qquad \tilde{g} = a\,, \\ \mathrm{SU}(2)_L &= \mathrm{SU}(2)_u\,, \qquad \widetilde{\mathrm{SU}(2)}_L = \mathrm{SU}(2)_v\,, \qquad \mathrm{SU}(2)_R^{\mathrm{diag}} = \mathrm{SU}(2)_w\,. \end{aligned} \tag{2.12}$$

- X_1':

$$\begin{aligned} g &= c\,, \qquad \tilde{g} = a^{-1}\,, \\ \mathrm{SU}(2)_L &= \mathrm{SU}(2)_u\,, \qquad \widetilde{\mathrm{SU}(2)}_L = \mathrm{SU}(2)_w\,, \qquad \mathrm{SU}(2)_R^{\mathrm{diag}} = \mathrm{SU}(2)_v\,. \end{aligned} \tag{2.13}$$

- X_2:

$$\begin{aligned} g &= a\,, \qquad \tilde{g} = b^{-1}\,, \\ \mathrm{SU}(2)_L &= \mathrm{SU}(2)_v\,, \qquad \widetilde{\mathrm{SU}(2)}_L = \mathrm{SU}(2)_u\,, \qquad \mathrm{SU}(2)_R^{\mathrm{diag}} = \mathrm{SU}(2)_w\,. \end{aligned} \tag{2.14}$$

- X_2':

$$\begin{aligned} g &= c^{-1}\,, \qquad \tilde{g} = b\,, \\ \mathrm{SU}(2)_L &= \mathrm{SU}(2)_v\,, \qquad \widetilde{\mathrm{SU}(2)}_L = \mathrm{SU}(2)_w\,, \qquad \mathrm{SU}(2)_R^{\mathrm{diag}} = \mathrm{SU}(2)_u\,. \end{aligned} \tag{2.15}$$

- X_3:

$$\begin{aligned} g &= a^{-1}\,, \qquad \tilde{g} = c\,, \\ \mathrm{SU}(2)_L &= \mathrm{SU}(2)_w\,, \qquad \widetilde{\mathrm{SU}(2)}_L = \mathrm{SU}(2)_u\,, \qquad \mathrm{SU}(2)_R^{\mathrm{diag}} = \mathrm{SU}(2)_v\,. \end{aligned} \tag{2.16}$$

- X_3':

$$\begin{aligned} g &= b\,, \qquad \tilde{g} = c^{-1}\,, \\ \mathrm{SU}(2)_L &= \mathrm{SU}(2)_w\,, \qquad \widetilde{\mathrm{SU}(2)}_L = \mathrm{SU}(2)_v\,, \qquad \mathrm{SU}(2)_R^{\mathrm{diag}} = \mathrm{SU}(2)_u\,. \end{aligned} \tag{2.17}$$

Consider next the homology three-cycles of these G_2 holonomy manifolds. Following [5], let $\hat{D}_i \subset \mathrm{SU}(2)^3$ be the i^{th} copy of SU(2) given by $(g_i, g_{i+1}, g_{i+2}) = (g, 1, 1)$, where the index i is defined mod 3. In $Y = \mathrm{SU}(2)^3/\,\mathrm{SU}(2)$, the $\hat{D}_i$ project to three-cycles D_i. Since $b_2(Y) = 2$, D_i are not independent and obey the relation

$$D_1 + D_2 + D_3 = 0\,. \tag{2.18}$$

The triality Σ_3 acts on the D_i by permutations. In terms of a, b, c, the D_i are given by

$$\begin{aligned} D_1 &:\ a = 1 = bc\,, \\ D_2 &:\ b = 1 = ca\,, \\ D_3 &:\ c = 1 = ab\,. \end{aligned} \tag{2.19}$$

It was argued in [5] that one obtains different G_2 holonomy manifolds depending on which three-cycle is "filled in". As an example, consider X_1 which is defined by filling in D_1, that is, D_1 is identified with the fiber $\mathbf{S}^3$. Indeed, it follows from $a = 1 = bc$ and (2.12) that $\sigma^a \neq 0$ while Σ^a vanish. The base $\mathbf{S}^3$ should correspond to the three-cycle of $b = 1$, namely D_2 because $\sigma^a = 0,\ \Sigma^a \neq 0$. In general we get

$$\begin{array}{ll} X_1 : (b^{-1}, a; D_2) & X_1' : (c, a^{-1}; -D_3) \\ X_2 : (a, b^{-1}; -D_1) & X_2' : (c^{-1}, b; D_3) \\ X_3 : (a^{-1}, c; D_1) & X_3' : (b, c^{-1}; -D_2)\,, \end{array} \tag{2.20}$$

where we label g,$\tilde{g}$ and Q the associative three-cycle (base). Recall that each $\mathbb{Z}_2$ acts by a parity transformation of X_i.

2.1 Quotients of $S(\mathbf{S}^3)$

We have seen that any quotient of these G_2 holonomy manifolds by a finite subgroup of the isometry preserves the G_2 structure. However, the triality Σ_3 of the base Y is broken by the quotient. As an example, consider a quotient by $\Gamma_1 \subset \mathrm{SU}(2)_u$. We impose the equivalence relation

$$b \sim b\gamma_1^{-1}\,, \qquad c \sim \gamma_1 c\,, \tag{2.21}$$

with $\gamma_1 \in \Gamma_1$. We will refer to this as a single quotient. The quotient leaves unbroken only the subgroup $\mathbb{Z}_2^{(1)}$. As noted in [5], this symmetry plays an important role in determining the curve describing the moduli space. We will mainly discuss the double quotient by

$$\Gamma_1 \subset \mathrm{SU}(2)_u\,, \qquad \Gamma_2 \subset \mathrm{SU}(2)_v\,. \tag{2.22}$$

The equivalence relation to be imposed is

$$a \sim \gamma_2 a\,, \qquad b \sim b\gamma_1^{-1}\,, \qquad c \sim \gamma_1 c \gamma_2^{-1}\,. \tag{2.23}$$

It follows that $Y_{\Gamma_1\times\Gamma_2}$, the quotient of Y, preserves none of the triality symmetry.

Let us consider the double quotient of $S(\mathbf{S}^3)$ in detail. We first discuss the homology of the base $Y_{\Gamma_1\times\Gamma_2}$. To this end, we start from the three-cycles D_i in Y and see how the quotient acts on them. D_3 projects to a cycle $D_3' = \mathbf{S}^3$. On the other hand, $D_i,\ i = 1, 2$ project to N_i-fold covers of $D_i' = \mathbf{S}^3/\Gamma_i$ with N_i the order of the finite group Γ_i. These cycles obey the relation

$$N_1 D_1' + N_2 D_2' + D_3' = 0\,. \tag{2.24}$$

Let us now discuss the topological structure of the double quotient of $S(\mathbf{S}^3)$. For instance, the quotient of $X_1 : (b^{-1}, a; D_2)$ is topologically

$$X_{1,\Gamma_1\times\Gamma_2} = \mathbb{R}^4/\Gamma_1 \times \mathbf{S}^3/\Gamma_2\,. \tag{2.25}$$

By definition, D_1' is filled in. It follows from (2.23) that the base manifold is homologous to D_2'. The topologies of the quotients are summarized by:

$$\begin{array}{ll} X_{1,\Gamma_1\times\Gamma_2} = \mathbb{R}^4/\Gamma_1 \times \mathbf{S}^3/\Gamma_2 : & \left(b^{-1}, a; D_2'\right) \\ X'_{1,\Gamma_1\times\Gamma_2} = \mathbb{R}^4/\Gamma_1 \times \mathbf{S}^3/\Gamma_2 : & \left(c, a^{-1}; -\dfrac{D_3'}{N_2}\right) \end{array}$$

$$
\begin{aligned}
X_{2,\Gamma_1\times\Gamma_2} &= \mathbb{R}^4/\Gamma_2 \times \mathbf{S}^3/\Gamma_1 : \qquad (a, b^{-1}; -D_1') \\
X_{2,\Gamma_1\times\Gamma_2}' &= \mathbb{R}^4/\Gamma_2 \times \mathbf{S}^3/\Gamma_1 : \qquad \left(c^{-1}, b; \frac{D_3'}{N_1}\right) \\
X_{3,\Gamma_1\times\Gamma_2} &= \mathbb{R}^4 \times \mathbf{S}^3/(\Gamma_1 \times \Gamma_2) : \qquad \left(a^{-1}, c; \frac{D_1'}{N_2}\right) \\
X_{3,\Gamma_1\times\Gamma_2}' &= \mathbb{R}^4 \times \mathbf{S}^3/(\Gamma_1 \times \Gamma_2) : \qquad \left(b, c^{-1}; -\frac{D_2'}{N_1}\right) .
\end{aligned} \tag{2.26}
$$

We will show later that M-theory on $X_{i,\Gamma_1\times\Gamma_2}$ and $X_{i,\Gamma_1\times\Gamma_2}'$ has the same vacuum structure, although the geometries are different from each other. A crucial point is that the vacuum structure can be specified by (i) the singularity of the geometry and (ii) the associative three-cycle. Since $X_{i,\Gamma_1\times\Gamma_2}$ and $X_{i,\Gamma_1\times\Gamma_2}'$ have the same properties, both correspond to the same point in the parameter space.

3. M-theory on double quotients

In this section we construct the curve describing the moduli space of M-theory on the double quotients defined by (2.23).

Let $\mathcal{N}_{\Gamma_1\times\Gamma_2}$ be the space of parameters of M-theory on the double quotients. Denote by $P_i \in \mathcal{N}_{\Gamma_1\times\Gamma_2}$ the points that correspond to the large volume limit of $X_{i,\Gamma_1\times\Gamma_2}, X_{i,\Gamma_1\times\Gamma_2}'$. The local coordinates of $\mathcal{N}_{\Gamma_1\times\Gamma_2}$ near these points are

$$
\begin{aligned}
\eta_1 &= \exp\left(\frac{2k}{3N_1} f_3 + \frac{k}{3N_1} f_1 + i\alpha_1'\right) , \\
\eta_2 &= \exp\left(\frac{2k}{3N_2} f_1 + \frac{k}{3N_2} f_2 + i\alpha_2'\right) , \\
\eta_3 &= \exp\left(\frac{2k}{3} f_2 + \frac{k}{3} f_3 + i\alpha_3'\right) ,
\end{aligned} \tag{3.1}
$$

with

$$
\alpha_i' = \int_{D_i'} C . \tag{3.2}
$$

Here k is a constant that can be determined using an instanton correction.

At P_i, we have

$$
(f_i, f_{i+1}, f_{i-1}) = \rho\,(-2, 1, 1) , \qquad \rho \to \infty , \tag{3.3}
$$

and thus

$$
\eta_i = e^{i\alpha_i'} , \qquad \eta_{i+1} = 0, \quad \eta_{i-1} = \infty . \tag{3.4}
$$

In order to determine the orders of the pole and zero we will use the membrane instanton corrections(for a discussion on membrane instantons, see e.g. [49]). One-instanton action is given by

$$
u = \exp\left(-V(Q) + i\int_Q C\right) , \tag{3.5}
$$

where Q is an associative three-cycle and $V(Q)$ is its volume.

Consider the different points:

- P_1: Let us first consider the large volume limit of $X_{1,\Gamma_1\times\Gamma_2}$. A non-trivial C-field that breaks the gauge symmetry can be turned on along an ALE fibration by applying fiberwise the duality between M-theory on K3 and heterotic string on T^3. In this case, we have

$$\int_{D_1'} C = \frac{2\pi\mu_1}{t_1}\,. \tag{3.6}$$

Note that this corresponds to a triple in the heterotic string dual [50]. Thus we find

$$\eta_1 = e^{2\pi i\mu_1/t_1}\,. \tag{3.7}$$

By comparing η_2, η_3 to u with $Q = D_2' = -\frac{1}{N_2}D_3'$, we find

$$\eta_2 \sim u^{+1}\,, \qquad \eta_3 \sim u^{-N_2}\,, \tag{3.8}$$

where we used $D_1' = 0$, which holds since this cycle is filled in. We now recall the ansatz of [5] that an M2-brane wrapping an associative three-cycle with $\int C = 2\pi\mu/t$ is a SYM instanton of $K_t \subset G_\Gamma$ with instanton number t. Here G_Γ is an ADE group corresponding to the finite group Γ. t is a positive integer that divides some of the Dynkin indices k_i of G_Γ. K_t is a subgroup of G_Γ that is left unbroken by the C-field, and defined by Dynkin indices whose elements consist of a set of integer k_i/t. Then a good local coordinate around P_1 should be taken to be $u^{1/t_1 h_{t_1}}$ which is equal to the gaugino bilinear condensate of SYM with the gauge group K_{t_1}. Here h_{t_1} is the dual Coxeter number of K_{t_1}. Therefore, we find that η_2 at P_1 has zeros of order $t_1 h_{t_1}$ and η_3 has poles of order $N_2 t_1 h_{t_1}$.

We next turn to the study of the large volume limit of $X_{2,\Gamma_1\times\Gamma_2}$. A non-trivial C-field on an ALE fibration is given by

$$\int_{D_1'} C = \frac{2\pi\mu_1}{t_1}\,. \tag{3.9}$$

The associative three-cycle is given by $Q = -D_3'/N_2 = +D_2'$. It is then easy to see that

$$\eta_1 = e^{2\pi i\mu_1/t_1}\,, \quad \eta_2 = 0^{t_1 h_{t_1}}\,, \quad \eta_3 = \infty^{N_2 t_1 h_{t_1}}\,, \tag{3.10}$$

which confirm that $X_{1,\Gamma_1\times\Gamma_2}$ and $X'_{1,\Gamma_1\times\Gamma_2}$ correspond to the same point in the parameter space.

- P_2: As in the case of P_1, one can turn on a C-field on D_2' so that

$$\eta_2 = e^{2\pi i\mu_2/t_2}\,. \tag{3.11}$$

η_3 has zeros of order $N_1 t_2 h_{t_2}$, and η_1 has poles of order $t_2 h_{t_2}$ since $Q = -D_1' = \frac{1}{N_1}D_3'$.

- P_3:

$$\eta_3 = 1\,, \quad \eta_1 \propto \exp\left(i\int_{D_1'} C\right), \quad \eta_2 \propto \exp\left(i\int_{D_2'} C\right). \tag{3.12}$$

Since the associative three-cycle is given by $Q = D_1'/N_2 = -D_2'/N_1$, we find that

$$\eta_1 = 0^{N_2}\,, \qquad \eta_2 = \infty^{N_1}\,. \tag{3.13}$$

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To summarize, the behaviour of η_i at the points P_i is given by:

	$P_1^{t_1,\mu_1}$	$P_2^{t_2,\mu_2}$	P_3
η_1	$e^{2\pi i\mu_1/t_1}$	$\infty^{t_2 h_{t_2}}$	0^{N_2}
η_2	$0^{t_1 h_{t_1}}$	$e^{2\pi i\mu_2/t_2}$	∞^{N_1}
η_3	$\infty^{N_2 t_1 h_{t_1}}$	$0^{N_1 t_2 h_{t_2}}$	1

(3.14)

3.1 The M-theory curve

Now we are ready to construct the curve describing the moduli space. One finds from the table (3.14) that η_i have the same order of zeros and poles. Thus, η_i can be regarded as meromorphic functions on a sphere. Let z be the coordinate of this sphere. η_i take the form

$$\eta_1 = c_1 \frac{(z-\gamma)^{N_2}}{\prod_{t_2,\mu_2}(z-\beta_{t_2,\mu_2})^{t_2 h_{t_2}}}\,, \tag{3.15}$$

$$\eta_2 = c_2 \frac{\prod_{t_1,\mu_1}(z-\alpha_{t_1,\mu_1})^{t_1 h_{t_1}}}{(z-\gamma)^{N_1}}\,, \tag{3.16}$$

$$\eta_3 = c_3 \frac{\prod_{t_2,\mu_2}(z-\beta_{t_2,\mu_2})^{N_1 t_2 h_{t_2}}}{\prod_{t_1,\mu_1}(z-\alpha_{t_1,\mu_1})^{N_2 t_1 h_{t_1}}}\,, \tag{3.17}$$

where $P_1^{t_1,\mu_1}, P_2^{t_2,\mu_2}, P_3$ are mapped to the points $z = \alpha_{t_1,\mu_1}, \beta_{t_2,\mu_2}, \gamma$, respectively. In addition we have to impose the relations

$$\begin{aligned} e^{2\pi i\mu_1/t_1} &= c_1 \frac{(\alpha_{t_1,\mu_1}-\gamma)^{N_2}}{\prod_{t_2,\mu_2}(\alpha_{t_1,\mu_1}-\beta_{t_2,\mu_2})^{t_2 h_{t_2}}}\,, \\ e^{2\pi i\mu_2/t_2} &= c_2 \frac{\prod_{t_1,\mu_1}(\beta_{t_2.\mu_2}-\alpha_{t_1,\mu_1})^{t_1 h_{t_1}}}{(\beta_{t_2,\mu_2}-\gamma)^{N_1}}\,, \\ 1 &= c_3 \frac{\prod_{t_2,\mu_2}(\gamma-\beta_{t_2.\mu_2})^{N_1 t_2 h_{t_2}}}{\prod_{t_1,\mu_1}(\gamma-\alpha_{t_1,\mu_{t_1}})^{N_2 t_1 h_{t_1}}}\,. \end{aligned} \tag{3.18}$$

From these one can see that

$$c_1^{N_1} c_2^{N_2} c_3 = (-1)^{N_1 N_2}\,, \tag{3.19}$$

where we used the formula $\sum_{t,\mu} t h_t = N$, with N being the order of the finite group. It thus follows that

$$\eta_1^{N_1}\eta_2^{N_2}\eta_3 = (-1)^{N_1 N_2}\,. \tag{3.20}$$

From this and (3.1), one finds that

$$N_1\alpha_1' + N_2\alpha_2' + \alpha_3' = N_1 N_2 \pi \mod 2\pi\,. \tag{3.21}$$

As in [5], we should be able to reproduce this relation using an anomaly argument. We will leave this as an open problem.

The existence of the curve shows that the moduli space of the double quotients interpolates smoothly between the three different classical geometries P_1, P_2, P_3. We can express the curve in terms of η_i by eliminating z. For simplicity, consider the case $\Gamma_1 = \mathbb{Z}_{N_1}, \Gamma_2 = \mathbb{Z}_{N_2}$. Then it is easy to verify that

$$\eta_2 = \eta_1^{-N_1/N_2} \left(\eta_1^{1/N_2} - e^{-2\pi i k_2/N_2} \right)^{N_1} , \tag{3.22}$$

with $k_2 = 0, 1, \ldots, N_2 - 1$. This can be rewritten in the form

$$\eta_1 = \left(1 - e^{2\pi i k_1/N_1} \eta_2^{1/N_1} \right)^{-N_2} , \tag{3.23}$$

with $k_1 = 0, 1, \ldots, N_1 - 1$. It is important to notice that the form of these curves is unique for any $\alpha, \beta, \gamma, c_i$ that obey (3.18).

3.2 The gauge theory

Consider now the low energy physics of M-theory on $X_{i,\Gamma_1\times\Gamma_2}$ (2.26). Consider $X_{1,\Gamma_1\times\Gamma_2} = \mathbb{R}^4/\Gamma_1 \times \mathbf{S}^3/\Gamma_2$. The singularity of $\mathbb{R}^4/\Gamma_1$ yields $\mathcal{N} = 1$ super Yang-Mills theory with the gauge group G_{Γ_1}, where G_{Γ_1} is the ADE group that corresponds to Γ_1. The gauge bosons in four dimensions arise from integrating field and wrapping M2-branes on the shrinking two-spheres of $\mathbb{R}^4/\Gamma_1$. We need to interpret the low-energy physics of the second quotient $\mathbf{S}^3/\Gamma_2$. A natural interpretation is that this quotient corresponds to the confining phase of $\mathcal{N} = 1$ SYM with the gauge group G_{Γ_2}. As argued in [4, 8, 9], an M2-brane wrapping a non-trivial one-cycle in $\mathbf{S}^3/\Gamma_2$ can be regarded as a QCD string of SYM. The domain wall that separates two distinct vacua is realized by an M5-brane wrapping the three-cycle $\mathbf{S}^3/\Gamma_2$. Since we get a product of two $\mathcal{N} = 1$ super Yang-Mills theories in different phases, we can deduce that in this scenario the dynamical scales of the two theories obey $\Lambda_1 \gg \Lambda_2$.

The low-energy physics of M-theory on $X_{2,\Gamma_1\times\Gamma_2} = \mathbb{R}^4/\Gamma_2 \times \mathbf{S}^3/\Gamma_1$ is the same with interchanging G_{Γ_1} and G_{Γ_2}.

Consider now $X_{3,\Gamma_1\times\Gamma_2} = \mathbb{R}^4 \times \mathbf{S}^3/(\Gamma_1 \times \Gamma_2)$. This quotient space is not singular and admits no normalizable zero modes. Thus, the low-energy physics of M-theory on $X_{3,\Gamma_1\times\Gamma_2}$ is interpreted as the confining phase of $G_{\Gamma_1} \times G_{\Gamma_1}$ SYM theory. One support for this comes from the fact that, say for $\Gamma_1 = \mathbb{Z}_{N_1}, \Gamma_2 = \mathbb{Z}_{N_2}$

$$\pi_1(X_{3,\Gamma_1\times\Gamma_2}) = \pi_1(X'_{3,\Gamma_1\times\Gamma_2}) = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} . \tag{3.24}$$

An M2-brane wrapping non-trivial one-cycles is a QCD string whose end point is a static quark that belongs to $(\mathbf{N}_1, \mathbf{N}_2)$ of $\mathrm{SU}(N_1) \times \mathrm{SU}(N_2)$.

The curve (3.22) and (3.23) exhibit spontaneous breaking of the discrete quantum R-symmetry of SYM. For instance, consider $\eta_1 \to e^{2\pi i}\eta_1$ in (3.22). A curve of label k_2 gets mapped to one with k_2+1. From (2.26), the phase of η_1 is identified with a vacuum angle of $\mathrm{SU}(N_2)$ SYM. Thus, the transformation $\eta_1 \to e^{2\pi i}\eta_1$ corresponds to a shift of the vacuum angle by 2π. From field theory viewpoint, this shift amounts to a change of the phase factor of a gaugino bilinear condensate, in agreement with our interpretation of the curve. We see that for each $k_2 = 0, 1, \ldots, N_2 - 1$, (3.22) describes a branch of the parameter space $\mathcal{N}_{\Gamma_1\times\Gamma_2}$ that interpolates between the UV and the confining phase of $\mathrm{SU}(N_2)$ SYM with a gaugino bilinear condensate labelled by k_2. Similarly, the curve (3.23) exhibits spontaneous breaking of discrete $\mathbb{Z}_{N_1}$ symmetry of $\mathrm{SU}(N_1)$ SYM.

Finally, note that since the different phases are smoothly interpolated, it may be regarded as a support of the picture of low energy dynamics of SYM: mass gap, confinement and chiral symmetry breaking [4, 5].

4. Superpotentials and membrane instantons

In this section we will compute the $\mathcal{N}=1$ superpotentials of M-theory on the G_2 holonomy quotients, and comment on membrane instantons.

4.1 Single quotient

We start with the single quotient with $\Gamma_2 = 1$. The M-theory curve reads [5]

$$\eta_2 = \eta_1^{-N_1} \prod_{t_1,\mu_1} \Big(\eta_1 - \exp(2\pi i \mu_1/t_1)\Big)^{t_1 h_{t_1}} , \qquad \eta_3 = \prod_{t_1,\mu_1} \Big(1 - \exp(-2\pi i \mu_1/t_1)\eta_1\Big)^{-t_1 h_{t_1}} , \tag{4.1}$$

and the structure of zeros and poles is given by:

$$\begin{array}{|c|c|c|c|} \hline & P_1^{t_1,\mu_1} & P_2 & P_3 \\ \hline \eta_1 & e^{2\pi i \mu_1/t_1} & \infty & 0 \\ \hline \eta_2 & 0^{t_1 h_{t_1}} & 1 & \infty^{N_1} \\ \hline \eta_3 & \infty^{t_1 h_{t_1}} & 0^{N_1} & 1 \\ \hline \end{array} \tag{4.2}$$

η_i are meromorphic functions on a sphere. Let z be the coordinate of this sphere. We map the points P_2, P_3 to $z = \omega, \omega^2$ respectively, with $\omega = e^{2\pi i/3}$. η_1 can be written in the form

$$\eta_1 = -\omega \frac{z-\omega^2}{z-\omega} . \tag{4.3}$$

In this parametrization the point $P_1^{t_1,\mu_1}$ corresponds to

$$z = \alpha_{t_1,\mu_1} = \omega \, \frac{e^{2\pi i \mu_1/t_1} + \omega^2}{e^{2\pi i \mu_1/t_1} + \omega} = \frac{\cos\left(\frac{\pi\mu_1}{t_1} + \frac{\pi}{3}\right)}{\cos\left(\frac{\pi\mu_1}{t_1} - \frac{\pi}{3}\right)} . \tag{4.4}$$

It follows from (4.1) that

$$\begin{aligned} \eta_2 &= c' \, \frac{\prod_{t_1,\mu_1} (z-\alpha_{t_1,\mu_1})^{t_1 h_{t_1}}}{(z-\omega^2)^{N_1}} , \\ \eta_3 &= c'' \, \frac{(z-\omega)^{N_1}}{\prod_{t_1,\mu_1} (z-\alpha_{t_1,\mu_1})^{t_1 h_{t_1}}} , \end{aligned} \tag{4.5}$$

where

$$c' = \prod_{t_1,\mu_1} (1 + \omega^{-1} e^{2\pi i \mu_1/t_1})^{t_1 h_{t_1}} , \qquad c'' = \prod_{t_1,\mu_1} (\omega + e^{2\pi i \mu_1/t_1})^{-t_1 h_{t_1}} . \tag{4.6}$$

Note that the single quotient preserves the subgroup $\mathbb{Z}_2^{(1)} \subset \Sigma_3$ that acts on η_i as

$$\mathbb{Z}_2^{(1)} : \quad (\eta_1, \eta_2, \eta_3) \to (\eta_1^{-1}, \eta_3^{-1}, \eta_2^{-1}) , \tag{4.7}$$

which is equal to the action $z \to 1/z$.

Consider now the superpotential. The quantum corrections to the superpotential come from instantons that have two fermionic zero modes. They correspond to euclidean membranes wrapping associative three-cycles. These contributions should vanish at P_i since P_i correspond to the large volume (classical gauge theory) limit. Thus, we conclude that W is proportional to

$$f(z) \equiv (z-\omega)(z-\omega^2) \prod_{t_1,\mu_1} (z-\alpha_{t_1,\mu_1})\,. \tag{4.8}$$

We assume that W has no other zeros. Consider, for instance, the behavior near the point P_1. We write

$$z \sim \alpha_{t_1,\mu_1} + \tilde{u}\,, \tag{4.9}$$

with $\tilde{u} = u^{1/t_1 h_{t_1}}$ being a good local coordinate around $P_1^{t_1,\mu_1}$.

The superpotential W behaves as

$$W \sim \tilde{u}\,, \tag{4.10}$$

which is consistent with the fact that $\tilde{u}$ has been identified with a gaugino bilinear condensate of SYM with gauge group K_{t_1}. So the above result is in accord with a field theoretic result.

Since $W(z)$ is a meromorphic function on the sphere, $W(z)$ has the same order of zeros and poles. Thus, it takes the form

$$W(z) = ic\,\frac{f(z)}{g(z)}\,, \tag{4.11}$$

where $g(z)$ is a polynomial of degree $l \equiv 2 + \sum_{t_1,\mu_1} 1$, and c is a constant.

In order to determine $g(z)$, we recall that the single quotient preserves the discrete symmetry $\mathbb{Z}_2^{(1)} \subset \Sigma_3$ that acts on the superpotential as R-symmetry

$$W(1/z) = -W(z)\,. \tag{4.12}$$

Using the relation

$$\prod_{\mu_1} -\alpha_{t_1,\mu_1} = 1 \qquad \text{for} \quad t_1 = 2,3,4,5\,, \tag{4.13}$$

we find that

$$f(1/z) = -z^{-l} f(z)\,. \tag{4.14}$$

Thus, g obeys

$$g(1/z) = z^{-l} g(z)\,. \tag{4.15}$$

When $\Gamma_1 = \mathbb{Z}_{N_1}$, W takes the form

$$W(z) = ic\,\frac{z^3-1}{z^3+\xi z^2+\xi z+1}\,. \tag{4.16}$$

Here ξ is a constant and $\xi(N_1 = 1) = 0$ in order to be consistent with [5]. We can rewrite the superpotential in terms of the couplings η_i, with manifest $\mathbb{Z}_2^{(1)}$ symmetry

$$W = -ic(\omega - \omega^{-1}) \frac{\eta_1 + \eta_2^{1/N_1} + \eta_3^{1/N_1} + \eta_1^{-1} + \eta_2^{-1/N_1} + \eta_3^{-1/N_1}}{(1-2\xi)\left(\eta_1 - \eta_1^{-1}\right) + (1+\xi)\left(\eta_2^{1/N_1} + \eta_3^{1/N_1} - \eta_2^{-1/N_1} - \eta_3^{-1/N_1}\right)}. \tag{4.17}$$

Let us study now the membrane instanton corrections to η_i, where we consider the case $\Gamma_1 = \mathbb{Z}_{N_1}$. We have

$$\frac{z^3 - 1}{z^3 + \xi z^2 + \xi z + 1} = u\,, \tag{4.18}$$

with u an instanton factor.

Let $z_1(u), z_2(u), z_3(u)$ be the solutions to this equation such that $z_i(u = 0) = \omega^{i+2}$. u in z_1 comes from a fractional M2 instanton of instanton number $1/N_1$. On the other hand, u in z_2, z_3 is due to an ordinary M2 instanton wrapping the associative three cycle $\mathbf{S}^3/\mathbb{Z}_{N_1}$. By plugging the solutions into (4.3) and (4.5), the coupling constants η_i can be written in terms of an power expansion in u. The instanton contributions to η_1 around P_1 take the form

$$\eta_1(z = z_1(u)) = 1 - \frac{2i(\xi+1)}{3\sqrt{3}}\,u - \frac{2(\xi+1)^2}{27}\,u^2 + \cdots\,. \tag{4.19}$$

M2 instanton contributions to η_2 take the form

$$\eta_2(z = z_2(u)) = 1 + \frac{N_1 i(\xi-2)}{3\sqrt{3}}\,u - \frac{N_1\Big((\xi-2)N_1 - 3\xi\Big)(\xi-2)}{54}\,u^2 + \cdots\,. \tag{4.20}$$

4.2 Double quotient

Consider the double quotient and let $\Gamma_1 = \mathbb{Z}_{N_1}, \Gamma_2 = \mathbb{Z}_{N_2}$. We have

$$\eta_1 = c_1\,\frac{(z-\gamma)^{N_2}}{(z-\beta)^{N_2}}\,, \qquad \eta_2 = c_2\,\frac{(z-\alpha)^{N_1}}{(z-\gamma)^{N_1}}\,, \qquad \eta_3 = c_3\,\frac{(z-\beta)^{N_1N_2}}{(z-\alpha)^{N_1N_2}}\,. \tag{4.21}$$

For simplicity, we take

$$\alpha = 1\,, \qquad \beta = \omega\,, \qquad \gamma = \omega^2\,. \tag{4.22}$$

As in the case of the single quotient, the superpotential must vanish at P_1, P_2, P_3. It is a meromorphic function on the sphere and therefore the order of its poles is three. Thus, it takes the form

$$W = ic\,\frac{z^3 - 1}{z^3 + \xi_1 z^2 + \xi_2 z + \xi_3}\,. \tag{4.23}$$

Note that the double quotient breaks $\mathbb{Z}_2^{(1)}$. As a check, consider the behavior around P_1 by setting

$$z \sim 1 + u\,. \tag{4.24}$$

The superpotential is given by

$$W \sim u\,. \tag{4.25}$$

Since $u \sim \Lambda_1^3$, this can be identified with the superpotential of $\mathrm{SU}(N_1)$ SYM. Indeed, it is consistent that no dependence of Λ_2^3 appears, since we are working in the regime $\Lambda_1 \gg \Lambda_2$.

5. Double quotients with Wilson lines

In this section we will make some comments and present some questions on Wilson lines and the M-theory moduli space. We start with M-theory on the singular G_2 holonomy manifold $X_{1,\Gamma_1\times\Gamma_2} = \mathbb{R}^4/\mathbb{Z}_{N_1} \times \mathbf{S}^3/\mathbb{Z}_{N_2}$. The low energy dynamics is described by a seven-dimensional $\mathrm{SU}(N_1)$ SYM compactified on $Q = \mathbf{S}^3/\mathbb{Z}_{N_2}$. As usual, supersymmetry is preserved by a twist, namely the identification of spin connections $w_{\mathrm{SO}(3)_R} = w_{\mathrm{SO}(3)_Q}$. The bosonic matter content of the resulting theory is found to be

$$\begin{array}{ccc} & \mathrm{SO}(1,3) & \mathrm{SO}(3) \\ A & \mathbf{4} & \mathbf{1} \\ \phi & \mathbf{1} & \mathbf{3} \end{array} \tag{5.1}$$

ϕ is a complex scalar which belongs to the adjoint representation of $\mathrm{SU}(N_1)$. Now we turn on Wilson lines for ϕ along a one-cycle γ of $\mathbf{S}^3/\mathbb{Z}_{N_2}$

$$g_\gamma = \mathrm{P}\exp i \oint_\gamma \phi = \mathrm{diag}(I_{n_1}, e^{2\pi i/N_2} I_{n_2}, \cdots, e^{2\pi i(k-1)/N_2} I_{n_k}), \quad k \le N_2\,, \tag{5.2}$$

with $\sum_i n_i = N_1$ (see also [45][51]). g_γ is an N_1-dimensional reducible representation of $\mathbb{Z}_{N_2}$. Recall that the Cartan part of ϕ is given by

$$\phi^a = \int_{\beta_a} C, \quad a = 1, 2, \ldots, N_1 - 1\,. \tag{5.3}$$

Here β_a are collapsing two-cycles in $\mathbb{R}^4/\mathbb{Z}_{N_1}$. Thus, the Wilson lines are given by integrating the C-field on three-cycles $C_a = \beta_a \times \gamma$.

The Wilson line breaks the gauge group $\mathrm{SU}(N_1)$ to $\prod_i \mathrm{SU}(n_i)$. The mass of the W-bosons is of order r_0^{-1}, where r_0 is the size of the one-cycle in $\mathbf{S}^3/\mathbb{Z}_{N_2}$. Alternatively, the seven-dimensional SYM has the interaction term

$$\int d^7x\, \mathrm{tr}[\phi, A]^2\ , \tag{5.4}$$

which gives mass to A. Since we are working in the low energy regime $E \ll r_0^{-1}$, the massive modes may be neglected and the gauge theories decouple from one another. In the context of M-theory, this Higgs mechanism can be interpreted as follows. The gauge bosons of $\mathrm{SU}(N_1)$ come from M2-branes wrapping collapsing cycles β in $\mathbb{R}^4/\mathbb{Z}_{N_1}$. The Wilson lines turn on a C field, which gives mass to some of the wrapped membranes. Consequently we have $\prod_i \mathrm{SU}(n_i)$ symmetry rather than $\mathrm{SU}(N_1)$.

Consider now $X_{2,\Gamma_1\times\Gamma_2}$ and $X_{3,\Gamma_1\times\Gamma_2}$ without turning on Wilson lines. As we have seen, these two geometries can be interpolated smoothly. It is unlikely that we interpolate between them and $X_{1,\Gamma_1\times\Gamma_2}$ with Wilson lines. A natural guess is that with a Wilson line we get a new branch of the moduli space that is disconnected with the branch without Wilson lines, and we view the Wilson lines as discrete moduli. If such a branch exists, it is interesting to know whether there is a G_2 flop that interpolates $X_{1,\Gamma_1\times\Gamma_2}$ with Wilson lines smoothly to a G_2 manifold that describes the vacuum structure of $\mathcal{N} = 1$ $\prod \mathrm{SU}(n_i)$ SYM.

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For the latter, the geometry should possess no ADE singularity because of a mass gap of SYM. It should also account for the Witten index of $\mathcal{N}=1$ $\prod \mathrm{SU}(n_i)$ SYM together with that of $\mathcal{N}=1$ $\mathrm{SU}(N_2)$ SYM

$$\mathrm{tr}\,(-1)^F = N_2 \prod n_i\,. \tag{5.5}$$

A candidate geometry is $X = \mathbb{R}^4 \times \{\mathbf{S}^3/(\mathbb{Z}_{n_1}\times\mathbb{Z}_{N_2})\#\cdots\#\ \mathbf{S}^3/(\mathbb{Z}_{n_k}\times\mathbb{Z}_{N_2})\}$, namely a "multi-center" G_2 holonomy manifold such that around each center it looks like $\mathbb{R}^4 \times \mathbf{S}^3/(\mathbb{Z}_{n_i}\times\mathbb{Z}_{N_2})$. $\mathbb{Z}_{N_2}$ acts diagonally. The number of discrete vacua can be seen by looking at a fractional change of C-field fluxes on associative three-cycles under a shift of the vacuum angle θ by 2π. Indeed,

$$\int_{\mathbb{Z}_{n_i}\backslash\, \mathbf{S}^3/\mathbb{Z}_{N_2}} C \to \int_{\mathbb{Z}_{n_i}\backslash\, \mathbf{S}^3/\mathbb{Z}_{N_2}} C + \frac{2\pi}{n_i N_2} \tag{5.6}$$

under the shift of the vacuum angle. We also find that $\pi_1(X) = \mathbb{Z}_{N_2}\times\mathbb{Z}_{n_1}\times\mathbb{Z}_{n_2}\times\cdots$. This is consistent with the interpretation that an M2-brane wrapping non-trivial one-cycles of X is a QCD string whose end point belongs to $(\mathbf{N_2}, \mathbf{n_1}, \mathbf{n_2}, \ldots)$ of $\mathrm{SU}(N_2)\times\prod \mathrm{SU}(n_i)$. Note also that KK reduction of this geometry along the Hopf fibers of $\mathbb{Z}_{n_i}\backslash\mathbf{S}^3/\mathbb{Z}_{N_2}$, yields a generalized conifold [52] that takes the form of multi small resolved conifolds with RR-fluxes on the Hopf bases $\mathbf{S}^2/\mathbb{Z}_{N_2}$. The mirror of this CY is reminiscent of a setup studied in [31] in the context of SYM with a product group via a geometric transition.

6. Discussion

In the paper we extended results of [5] to more general quotients of $S(\mathbf{S}^3)$, and in particular the "double" quotient, where the quotient group is a subgroup of the isometry of $S(\mathbf{S}^3)$. We analysed the moduli space of M-theory on the quotient and show that the moduli space interpolates smoothly between three different types of classical geometries. These three geometries correspond to different low energy dynamics. We computed the exact superpotentials of M-theory on the quotients of $S(\mathbf{S}^3)$, and discussed the membrane instanton physics. We commented on turning on Wilson lines and breaking of the gauge symmetry and discussed some of the implications.

There are several issues that deserve further study. One is to derive equation (3.21) by an anomaly analysis. Another issue is to understand in the context of M-theory dynamics the singularities of the superpotentials computed in section 4. It would also be interesting to know the precise relation between the complexified gauge coupling of SYM τ and η_i, $\tau=\tau(\eta_i)$. Once given, the analysis of section 4 will provide us with information about nonperturbative corrections to the $\mathcal{N}=1$ SYM coupling.

Finally, the structure of the moduli space of M-theory on the G_2 holonomy manifolds with Wilson lines is still largely unclear and deserves further study. From the discussion of the previous section 5, one may deduce a duality of superstrings, which is an extension of [12]. The KK reduction of $\mathbb{R}^4/\mathbb{Z}_{N_1}\times\mathbf{S}^3/\mathbb{Z}_{N_2}$ along an $\mathbf{S}^1$ in the ALE fibration leads to N_1 D6 branes wrapping $\mathbf{S}^3/\mathbb{Z}_{N_2}$ of a deformed conifold. On the other hand, from KK reduction of $X = \mathbb{R}^4 \times \{\mathbf{S}^3/(\mathbb{Z}_{n_1}\times\mathbb{Z}_{N_2})\#\cdots\#\ \mathbf{S}^3/(\mathbb{Z}_{n_k}\times\mathbb{Z}_{N_2})\}$, along the Hopf fibers of

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$\mathbb{Z}_{n_i}\backslash\mathbf{S}^3/\mathbb{Z}_{N_2}$, one obtains superstring theory on the resultant CY. We may conjecture that both are dual to each other. One can try to verify the duality in the context of topological strings [53]. The superstring theory with the D6 branes wrapping the three-cycle of a deformed conifold reduces to a topological open string that is given by SU(N_1) Chern-Simons theory on $\mathbf{S}^3/\mathbb{Z}_{N_2}$ with Wilson lines. It would be nice to see if this theory is dual to a topological closed string on the CY:

$$\begin{array}{ccc}
\begin{array}{c}\mathbb{R}^4/\mathbb{Z}_{N_1}\times\mathbf{S}^3/\mathbb{Z}_{N_2}\\ \text{with Wilson lines}\end{array} & \overset{\text{G}_2\text{ flop}}{\Longleftrightarrow} & X \\
\Downarrow & & \Downarrow \\
\begin{array}{c}N_1\text{ D6 branes}\\ \text{wrapping a three-cycle of a deformed conifold}\\ \text{with Wilson lines}\end{array} & \overset{\text{dual}}{\Longleftrightarrow} & \text{superstring on CY} \\
\Downarrow & & \Downarrow \\
\begin{array}{c}\text{topological open string}\\ \text{SU}(N_1)\text{ CS on }\mathbf{S}^3/\mathbb{Z}_{N_2}\\ \text{with Wilson lines}\end{array} & \overset{\text{dual}}{\Longleftrightarrow} & \text{topological closed string on CY}
\end{array}$$

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References

[1] G. Papadopoulos and P.K. Townsend, *Compactification of $D = 11$ supergravity on spaces of exceptional holonomy*, *Phys. Lett.* **B 357** (1995) 300 [hep-th/9506150].

[2] B.S. Acharya, *M theory, joyce orbifolds and super Yang-Mills*, *Adv. Theor. Math. Phys.* **3** (1999) 227 [hep-th/9812205].

[3] B.S. Acharya, *On realising $N = 1$ super Yang-Mills in m theory*, hep-th/0011089.

[4] M. Atiyah, J. Maldacena and C. Vafa, *An M-theory flop as a large-N duality*, *J. Math. Phys.* **42** (2001) 3209 [hep-th/0011256].

[5] M. Atiyah and E. Witten, *M-theory dynamics on a manifold of G_2 holonomy*, hep-th/0107177.

[6] E. Witten, *Anomaly cancellation on G_2 manifolds*, hep-th/0108165.

[7] B. Acharya and E. Witten, *Chiral fermions from manifolds of G_2 holonomy*, hep-th/0109152.

[8] B.S. Acharya, *Confining strings from G_2-holonomy spacetimes*, hep-th/0101206.

[9] B.S. Acharya and C. Vafa, *On domain walls of $N = 1$ supersymmetric Yang-Mills in four dimensions*, hep-th/0103011.

[10] R.L. Bryant and S. Salamon, *On the construction of some complete metrics with exceptional holonomy*, *Duke Math. J.* **58** (1989) 829.

JHEP04(2002)001

[11] G.W. Gibbons, D.N. Page and C.N. Pope, *Einstein metrics on S^3 $\mathbb{R}^3$ and $\mathbb{R}^4$ bundles*, *Commun. Math. Phys.* **127** (1990) 529.

[12] C. Vafa, *Superstrings and topological strings at large-N*, *J. Math. Phys.* **42** (2001) 2798 [hep-th/0008142].

[13] B.S. Acharya, J.P. Gauntlett and N. Kim, *Fivebranes wrapped on associative three-cycles*, *Phys. Rev.* **D 63** (2001) 106003 [hep-th/0011190].

[14] H. Nieder and Y. Oz, *Supergravity and D-branes wrapping special lagrangian cycles*, *J. High Energy Phys.* **03** (2001) 008 [hep-th/0011288].

[15] J.P. Gauntlett, N. Kim and D. Waldram, *M-fivebranes wrapped on supersymmetric cycles*, *Phys. Rev.* **D 63** (2001) 126001 [hep-th/0012195].

[16] F. Cachazo, K.A. Intriligator and C. Vafa, *A large-N duality via a geometric transition*, *Nucl. Phys.* **B 603** (2001) 3 [hep-th/0103067].

[17] J. Gomis, *D-branes, holonomy and M-theory*, *Nucl. Phys.* **B 606** (2001) 3 [hep-th/0103115].

[18] J.D. Edelstein and C. Nunez, *D6 branes and M-theory geometrical transitions from gauged supergravity*, *J. High Energy Phys.* **04** (2001) 028 [hep-th/0103167].

[19] S. Kachru and J. McGreevy, *M-theory on manifolds of G_2 holonomy and type IIA orientifolds*, *J. High Energy Phys.* **06** (2001) 027 [hep-th/0103223].

[20] J.D. Edelstein, K. Oh and R. Tatar, *Orientifold, geometric transition and large-N duality for SO/Sp gauge theories*, *J. High Energy Phys.* **05** (2001) 009 [hep-th/0104037].

[21] J. Gutowski and G. Papadopoulos, *Moduli spaces and brane solitons for M theory compactifications on holonomy G_2 manifolds*, *Nucl. Phys.* **B 615** (2001) 237 [hep-th/0104105].

[22] P. Kaste, A. Kehagias and H. Partouche, *Phases of supersymmetric gauge theories from M-theory on G_2 manifolds*, *J. High Energy Phys.* **05** (2001) 058 [hep-th/0104124].

[23] Y. Konishi and M. Naka, *Coset construction of Spin(7), G_2 gravitational instantons*, *Class. and Quant. Grav.* **18** (2001) 5521 [hep-th/0104208].

[24] M. Aganagic, A. Klemm and C. Vafa, *Disk instantons, mirror symmetry and the duality web*, *Z. Naturforsch.* **A57** (2002) 1–28 [hep-th/0105045].

[25] M. Aganagic and C. Vafa, *Mirror symmetry and a G_2 flop*, hep-th/0105225.

[26] M. Cvetič, G.W. Gibbons, H. Lu and C.N. Pope, *Supersymmetry M3-branes and G_2 manifolds*, *Nucl. Phys.* **B 620** (2002) 3 [hep-th/0106026].

[27] A. Brandhuber, J. Gomis, S.S. Gubser and S. Gukov, *Gauge theory at large-N and new G_2 holonomy metrics*, *Nucl. Phys.* **B 611** (2001) 179 [hep-th/0106034].

[28] R. Hernandez, *Branes wrapped on coassociative cycles*, *Phys. Lett.* **B 521** (2001) 371 [hep-th/0106055].

[29] M. Cvetič, G.W. Gibbons, J.T. Liu, H. Lu and C.N. Pope, *A new fractional D2-brane, G_2 holonomy and t-duality*, hep-th/0106162.

[30] M. Cvetič, G. Shiu and A.M. Uranga, *Three-family supersymmetric standard like models from intersecting brane worlds*, *Phys. Rev. Lett.* **87** (2001) 201801 [hep-th/0107143]; *Chiral four-dimensional $N = 1$ supersymmetric type IIA orientifolds from intersecting D6-branes*, *Nucl. Phys.* **B 615** (2001) 3 [hep-th/0107166].

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[31] F. Cachazo, S. Katz and C. Vafa, *Geometric transitions and $N = 1$ quiver theories*, hep-th/0108120.

[32] K. Sugiyama and S. Yamaguchi, *Cascade of special holonomy manifolds and heterotic string theory*, *Nucl. Phys.* **B 622** (2002) 3 [hep-th/0108219].

[33] M. Cvetič, G.W. Gibbons, H. Lu and C.N. Pope, *Cohomogeneity one manifolds of Spin(7) and G_2 holonomy*, hep-th/0108245.

[34] J. Gutowski, *Stringy domain walls of $N = 1$, $D = 4$ supergravity*, *Nucl. Phys.* **B 627** (2002) 381 [hep-th/0109126].

[35] F. Cachazo, B. Fiol, K.A. Intriligator, S. Katz and C. Vafa, *A geometric unification of dualities*, hep-th/0110028.

[36] A. Giveon, A. Kehagias and H. Partouche, *Geometric transitions, brane dynamics and gauge theories*, hep-th/0110115.

[37] M. Aganagic and C. Vafa, *G_2 manifolds, mirror symmetry and geometric engineering*, hep-th/0110171.

[38] M. Cvetič, G.W. Gibbons, H. Lu and C.N. Pope, *Orientifolds and slumps in G_2 and Spin(7) metrics*, hep-th/0111096.

[39] S. Frolov and A.A. Tseytlin, *$\mathbb{R}^4$ corrections to conifolds and G_2 holonomy metrics*, hep-th/0111128.

[40] G. Curio, B. Kors and D. Lüst, *Fluxes and branes in type II vacua and M-theory geometry with G_2 and Spin(7) holonomy*, hep-th/0111165.

[41] A. Bilal, J.-P. Derendinger and K. Sfetsos, *(Weak) G_2 holonomy from self-duality, flux and supersymmetry*, hep-th/0111274.

[42] M. Cvetič, G.W. Gibbons, H. Lu and C.N. Pope, *M-theory conifolds*, *Phys. Rev. Lett.* **88** (2002) 121602 [hep-th/0112098].

[43] A. Brandhuber, *G_2 holonomy spaces from invariant three-forms*, hep-th/0112113.

[44] M. Cvetič, G.W. Gibbons, H. Lu and C.N. Pope, *A G_2 unification of the deformed and resolved conifolds*, hep-th/0112138.

[45] E. Witten, *Deconstruction, G_2 holonomy and doublet-triplet splitting*, hep-ph/0201018.

[46] S.A. Hartnoll, *Dirac equation on a G_2 manifold*, hep-th/0201271.

[47] R. Hernandez and K. Sfetsos, *An eight-dimensional approach to G_2 manifolds*, hep-th/0202135.

[48] S.L. Shatashvili and C. Vafa, *Superstrings and manifold of exceptional holonomy*, hep-th/9407025;
T. Eguchi and Y. Sugawara, *CFT description of string theory compactified on non-compact manifolds with G_2 holonomy*, *Phys. Lett.* **B 519** (2001) 149 [hep-th/0108091];
R. Blumenhagen and V. Braun, *Superconformal field theories for compact G_2 manifolds*, *J. High Energy Phys.* **12** (2001) 006 [hep-th/0110232];
R. Roiban and J. Walcher, *Rational conformal field theories with G_2 holonomy*, *J. High Energy Phys.* **12** (2001) 008 [hep-th/0110302];
T. Eguchi and Y. Sugawara, *String theory on G_2 manifolds based on gepner construction*, hep-th/0111012;

B. Noyvert, *Unitary minimal models of* $\mathrm{SW}(3/2, 3/2, 2)$ *superconformal algebra and manifolds of* G_2 *holonomy*, *J. High Energy Phys.* **03** (2002) 030 [hep-th/0201198].

[49] J.A. Harvey and G.W. Moore, *Superpotentials and membrane instantons*, hep-th/9907026.

[50] J. de Boer et al., *Triples, fluxes and strings*, *Adv. Theor. Math. Phys.* **4** (2002) 995 [hep-th/0103170].

[51] R. Gopakumar and C. Vafa, *Branes and fundamental groups*, *Adv. Theor. Math. Phys.* **2** (1998) 399 [hep-th/9712048].

[52] S. Gubser, N. Nekrasov and S. Shatashvili, *Generalized conifolds and four dimensional* $N = 1$ *superconformal theories*, *J. High Energy Phys.* **05** (1999) 003 [hep-th/9811230].

[53] R. Gopakumar and C. Vafa, *On the gauge theory/geometry correspondence*, *Adv. Theor. Math. Phys.* **3** (1999) 1415 [hep-th/9811131].

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